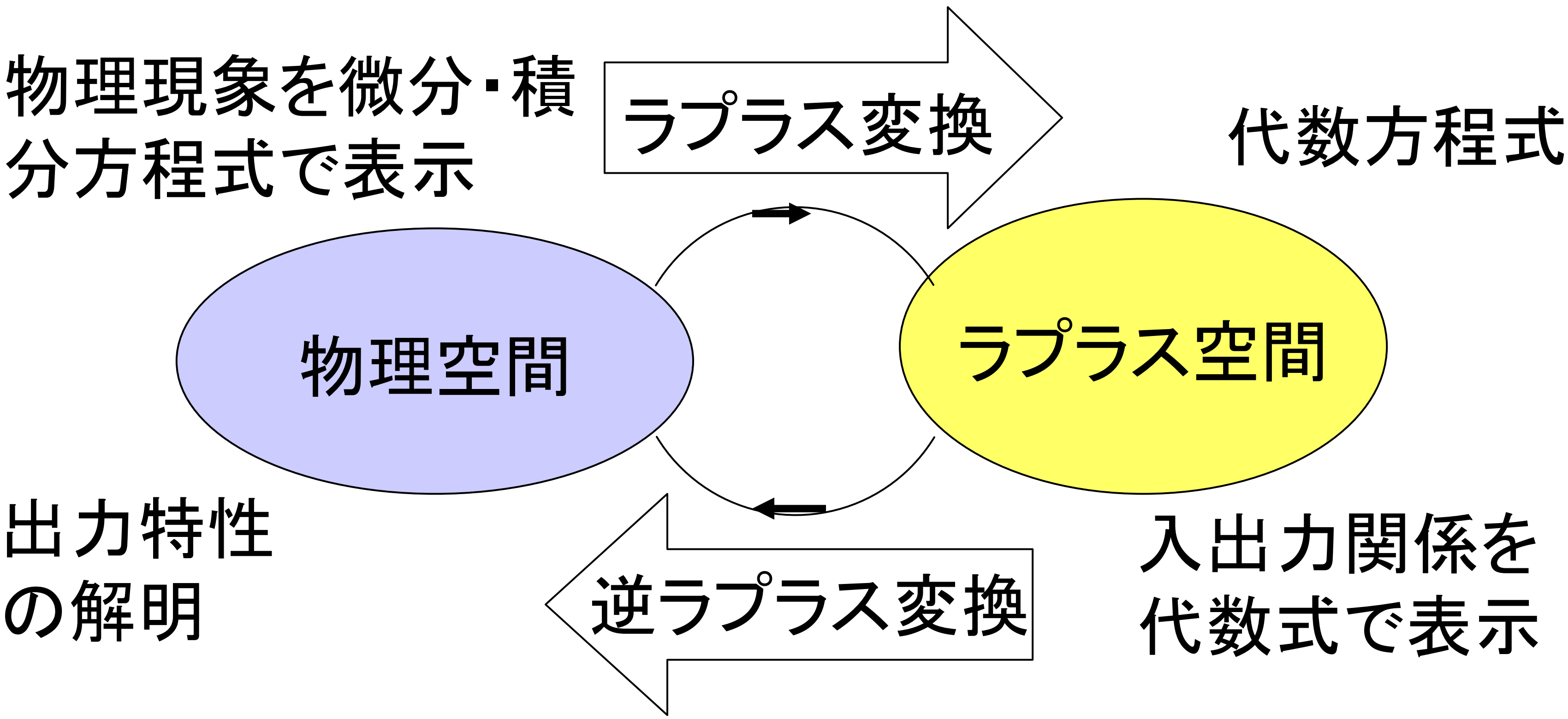
ラプラス変換（Laplace transformation）

ラプラス変換　：

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = F(s)$$

逆ラプラス変換:

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\, e^{st}\, ds = f(t)$$



[定理]

(1) $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$$\mathcal{L}\left\{\frac{d^k f}{dt^k}\right\} = s^k F(s) - \left[s^{k-1} f(0) + \cdots + f^{(k-1)}(0)\right]$$

(2) $\mathcal{L}\left\{\int_a^t f(t)\,dt\right\} = \frac{1}{s} F(s) + \frac{1}{s}\int_a^0 f(t)\,dt$

(3) $\mathcal{L}\left\{e^{-at} f(t)\right\} = F(s)_{s \to s+a}$

(4) $\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$; $u(t)$: step function

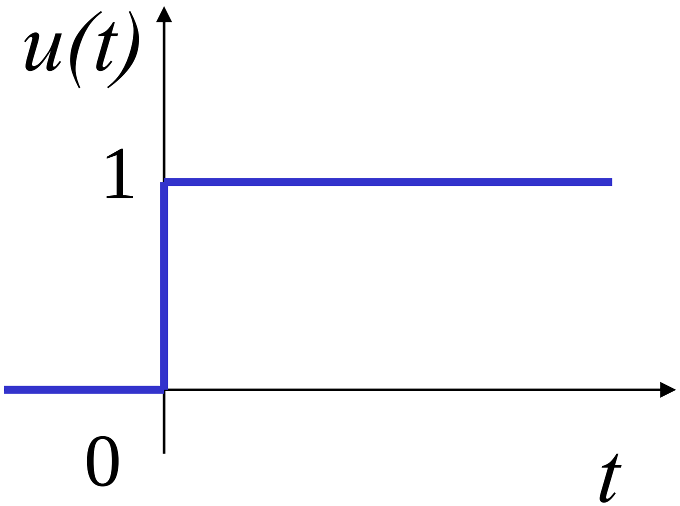
(5) $\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t) = f(0)$

(6) $\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$

［公式］

(1) $\mathcal{L}\{u(t)\} = \dfrac{1}{s};$

$u(t)$: step function



(2) $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$

(3) $\mathcal{L}\{t^{k}\} = \dfrac{k!}{s^{k+1}};\quad k$: integer

$$\sin bt = \frac{e^{ibt} - e^{-ibt}}{2i}, \quad \cos bt = \frac{e^{ibt} + e^{-ibt}}{2}$$

（物理系の例）

支配方程式:

$$a_2 x_o'' + a_1 x_o' + a_0 x_o = b_0 X_i e^{st}$$

入力: $x_i = X_i e^{st}$　　出力: $x_o = X_o e^{st}$

ラプラス変換

$$X_o(s) = \left(\frac{b_o}{a_2 s^2 + a_1 s + a_o} \right) X_i = G(s) X_i$$

逆ラプラス変換

$$x_o(t) = X_o(s) e^{st} = G(s) X_i e^{st}$$

逆ラプラス変換

$$X(s) = \frac{s+a}{(s+b)^n} = \sum_{k=1}^{n} \frac{C_k}{(s+b)^k}$$

$$C_k = \lim_{x \to -b} \left[\frac{1}{k!} \frac{d^k}{ds^k} \left\{ (s+b)^n X(s) \right\} \right]$$

例

$$X(s) = \frac{s+a}{s(s+b)^2(s+c)}$$

$$= \frac{C_1}{s} + \frac{C_2}{(s+b)^2} + \frac{C_3}{s+b} + \frac{C_4}{s+c}$$

$$C_1 = \lim_{s \to 0}\left[sX(s)\right] = \frac{a}{b^2 c}$$

$$C_4 = \lim_{s \to -c}\left[(s+c)X(s)\right] = \frac{a-c}{-c(b-c)^2}$$

$$C_2 = \lim_{s \to -c}\left[(s+b)^2 X(s)\right] = \frac{a-b}{-b(c-b)}$$

$$C_3 = \lim_{s \to -c}\left[\frac{d}{ds}\left\{(s+b)^2 X(s)\right\}\right] = \frac{b^2 - 2ab + ac}{-b^2(c-b)^2}$$

$$x(t) = L^{-1}\left\{X(s)\right\} = C_1 + C_2 t e^{-bt} + C_3 e^{-bt} + C_4 e^{-ct}$$