In prase of laziness: Categorical non-foundation of mathematics

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‘I want to say, in all seriousness, that a great deal of harm is being done in the modern world by belief in the virtuousness of work, and that the road to happiness and prosperity lies in an organized diminution of work.’

Bertrand Russell, “In Praise of Idleness”, 1935.

Varieties of normality

One way to think about the meaning of a λ term is by taking it to be its normal form. Then, terms with no normal form have no meaning, and therefore are all identified.

In a way, a term is thought to be normal if it cannot be rewritten any further, i.e., cannot be reduced to any other term. A λ term can be rewritten if it contains beta redex (reducible expressions), a subterm of the form $(\lambda x.M)N$, which can be reduced to $M[N/x]$ by the beta reduction rule.

However, there are other conceptions of normality. One can think of a term as normal if it does not contain a head redex, redex that appears in the head position of a term. Such a term is called in *head normal form*. This interpretation is called standard.

Varieties of normality

Yet another normality is *weak head normal form*. A term is in weak head normal form if it is an abstraction, namely, of the form $\lambda x.M$

This may sound too loose a conception of normality, but Samson Abramsky showed that this could be quite reasonable (Abramsky, "The lazy lambda calculus", 1987). For one thing, no functional programming language actually evaluates terms to its normal forms or head normal forms. For another, the domain semantics for the lazy lambda calculus has the non-trivial initial solution, that is, not the one-point domain.

Meaning as normal form

If we take the meaning of a term to be its weak head normal form, then any term is meaningful if only it is reduced to an abstraction.

Example. Let Ω be $(\lambda x.xx)(\lambda x.xx)$. Then, $\lambda x.\Omega$ does not have head normal form, but is in weak head normal form. So, in the lazy interpretation, Ω and $\lambda x.\Omega$ are not identified, although, in the standard interpretation, they are.

What is laziness good for?

Usual functional programming languages are lazy in the sense that they do not evaluate inner redexes (redexes that are not in the head position) or redexes within abstraction. This feature allow us to easily handle data structures such as stream (roughly, list of infinite length. More precisely, stream is the dual notion of list, in the sense that coalgebra is the dual of algebra).

Cf. Kees Doets and Jan van Eijck, *The Haskell Road To Logic, Maths And Programming*, 2004.

Stream and corecursion

A stream is a data structure defined, for example, as follows:

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One := (1, One)
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This definition looks circular, and it *is* circular. If you are *eager* to finish the evaluation, you will end up with running into an endless loop. You have to be *lazy* and refrain from evaluating it completely.

This kind of definition is called *corecursion*.

Top-down versus bottom-up

Greg Restall once said in a talk that, while semantics is bottom-up in nature, syntax is top-down. When engaged in syntax, we do not care about the details of subformulas but only about the structures of formulas or of derivations.

Based on this intuition, Restall introduced the notion of 'coformula', the dual of formula.

Top-down versus bottom-up

Abstract mathemaitcs is also carried out in the spirit of the top-down, without regard to the particular properties of individual objects (cf. Michael Makkai, "Towards a Categorical Foundation of Mathematics", 1998).

In the argument against Geoffrey Hellman, Steve Awodey also emphasises the top-down nature of mathematics.

Awodey's recommendation to philosophers of mathematics (esp. to structuralists)

Awodey, "Structure in mathematics and logic: A categorical perspective" (1996)

- Category theory is particularly suitable for studying mathematical structures.
- Nevertheless, structuralists have not paid due attention to category theory.
- Why not use category theory as a conceptual framework for thinking about mathematical structure?

Hellman's rejection

Hellman, "Does category theory provide a framework for mathematical structuralism?" (2003)

- Category theory is inadequate as a foundation of structuralism.
- The axioms of category theory is not 'assertory' but 'definitional'.
- Category theory does not answer the question about the 'home address' of its objects ("where do categories come from and where do they live?"), which an adequate structuralist foundation should.

Awodey's rebuttal

Awodey, 'An answer to Hellman's question: "Does category theory provide a framework for mathematical structuralism?" ' (2004)

- Hellman is too foundationalistic.
- Mathematics is in nature schematic and top-down.

Why is it hard for philosophers to follow Awodey's recommendation?

Benacerraf's dilemma

Paul Benacerraf, "Mathematical truth", 1983.

It is impossible to achieve the following two goals which have motivated philosophers' attempts to account for mathematical truth.

- To have a homogeneous semantics for the mathematical language and the rest of our linguistic activity.
- To have a homogeneous epistemology for the mathematical knowledge and the rest of our knowledge.

For Benacerraf, such semantics should be 'referential', that is, model-theoretic or truth-conditional. He assumes that "truth conditions for the language (e. g., English) to which mathematese appears to belong are to be elaborated much along the lines that Tarski articulated" (p. 410).

Austin's complaint

Austin, "Performative utterances", in *Philosophical Papers*, 1979.

Most philosophers tend to think that ' the sole interesting business, of any utterance–that is, of anything we say–is to be true or at least false' (p. 233).

Foundation versus Non-Foundation

Foundationalist	Non-Foundationalist
Set Theory	Category Theory
Bottom-up	Top-down
Eager	Lazy
Assertion	Definition
Referential Utterance	Performative Utterance
Implementation	Interface
Recursion	Corecursion
Individualism	Structuralism
Logicism	Axiomatism
Iterative Conception of Sets	Typical Ambiguity
Concrete	Abstract

These are two different but supplementary aspects of our linguistic activities, including mathematical and computer-scientific activities.

We need both.

"A formal system standing alone is an incomplete entity: it needs its interpretation."

Dana Scott, " Rules and derived rules ", 1974.

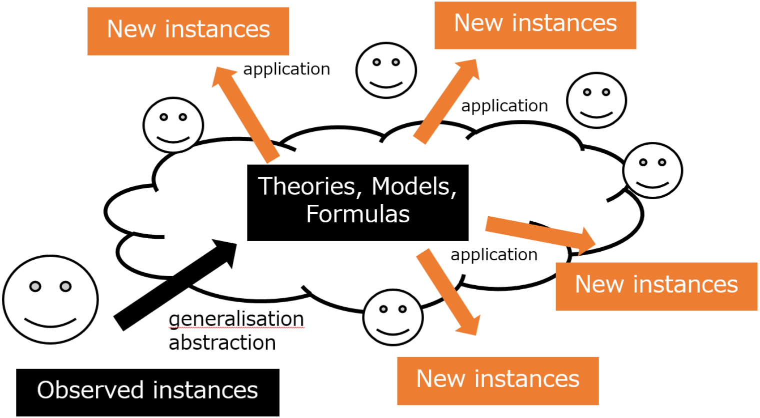
Feferman's distinction of axioms

Solomon Feferman, "Does mathematics need new axioms?", 1999.

Feferman distinguishes two kinds of axioms: 'foundational' axioms and 'structural' ones. The former assert obvious truth about certain well-known mathematical entities such as sets or natural numbers.

The latter have the function of defining a new class of mathematical structures. According to Feferman, the value of these axioms lies in its ability to organise mathematician's work and to 'package and communicate our knowledge in digestible way'.

What mathematics does



What is mathematics all about?

Mathematics is about literally *anything* that satisfies a set of axioms, not confined to so called mathematical objects or mathematical structures. Mathematical statements can be applied (though approximately) to things in the real world such as pebbles, planets, particles, water, human behaviours, strategies for chess, moves of the Rubik's Cube, and so on.

For me, it is one of the most irrational idea that mathematics is all about sets or any other particular kinds of mathematical entities whatsoever.

What is mathematics at all?

Andy Clark claims in his *Natural-Born Cyborgs* (2003) that languages, among other artefacts, are part of our cognitive faculties.

Similarly, we can say that mathematics is part of our cognitive, and communicative, faculties. Arguably, mathematics has been the most effective and reliable instrument for knowledge acquisition and information communication, at least so far.

Tanimura's tweet on physical theories

TANIMURA Shogo @tani6s · 2018年4月3日

私は記事「量子論と代数—思考と表現の進化論」とその補足解説で、現実の現象を説明し精度よく予測するという意味での物理理論の成功、とくに数学的記述の効用は、奇跡ではなく、成功したものがより多くのコピーを残す淘汰の結果だと論じました。

'I argued [...] that the success of physical theory in the sense of explaining and accurately predicting real phenomena, in particular the utility of mathematical descriptions, is not a miracle, but the result of more successful ones being selected and thus leaving more copies.'

Exactly the same should be said of the mathematics as a whole, or even of science as a whole.

Theft?

'The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil.'

Russell, *Introduction to Mathematical Philosophy*, 1919.

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