A non-model theoretic realism in mathematics*

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1 Introduction

Why are many mathematicians realists, while many philosophers are not? This question becomes even more striking when one considers the fact that modern mathematics is carried out totally in the axiomatic manner. The main aim of this article is to give an answer to this question.

One reason lies in different conceptions of reality between philosophers and mathematicians. For example, Abelson and Sussman wrote

> [a computational process] cannot be seen or touched. It is not composed of matter at all. However, it is very real. It can perform intellectual work. It can answer questions. It can affect the world by disbursing money at a bank or by controlling a robot arm in a factory. (Abelson and Sussman, [1], p.1)

They think that a computer process is real because it has certain effects and functions. On the other hand, many philosophers seem to think that in order for something to be real,

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it must exist in space-time; it must be independent of our senses, perceptions, thoughts, etc.; it must have causal effects, etc. Let us call the reality in the former sense *functional*, and that in the latter *substantial*. When philosophers are discussing reality, it is often a substantial one that is intended. However we certainly use the word "reality" in many senses, and there is no reason to confine the word in any one of them, or to regard any one of them as central or normal. It is clear that mathematical objects are not substantially real. One may be led to think that the question about the reality of mathematics is only concerned with how we define the word "reality," and that mathematics is not substantially real, but functionally real. However this is not the case. The question of reality has a certain significance within mathematics.

Abelson and Sussman demarcate computation from mathematics, and do not extend their view to mathematics in general. However, the development of intuitionistic mathematics over the past century has shown that a fairly large part of mathematics can be viewed as computational. How far we can extend this view, i.e., to what extent mathematics can be thought of as (functionally) real is an interesting question. Therefore reality has it own significance apart from philosophical or metaphysical discussion.

A difference between computation and mathematics is that the former makes more of processes, while the latter makes more of results. For example, in computation, a function is viewed as a program that prescribes how to process one piece of data into another. In mathematics, however, it is viewed as a set of ordered pairs of inputs and outputs, no matter how the outputs are obtained. In other words, a function is reified and seen as a substantial object in mathematics. This, I think, is a major source of the controversy over realism in mathematics.

Most philosophical arguments against realism seem to be based, for one thing, on the non-existence of mathematical objects such as numbers, sets, etc. and, for another, on the irrelevance of the state of actual affairs to the truth and falsehood of a mathematical sentence. Thus, many realist replies consist in claiming some other form of existence than that of ordinary perceptible objects, and then associating with it the truth of a statement.

Such realists look into the realm of the thoughts, forms etc. for mathematical objects to which mathematical statements refer. In doing so, they are tacitly employing a form of model-theoretic interpretation for mathematical statements.

This is understandable and even natural, because the model-theoretic interpretation is so dominant in current logic that we are quite accustomed to interpreting formulas or statements by means of models. However, this is not the only way of interpreting mathematical language. One alternative is the operational interpretation, where the notion of truth is redundant and hardly seems to demand that mathematical statement refer to something outside mathematics. The meaning of an expression is fully determined by the formal system, and the notion of truth is replaced by the notion of provability or derivability. It might at first seem that realism is out of place when one is employing this interpretation. I shall argue that some form of realism can still be maintained in the alternative.

My conclusion is that objects of mathematics are procedures prescribed by mathematical terms, formulas and proofs, and they can be viewed as real by weaker criteria than those of philosophers. I formulate the followings as criteria employed by mathematicians: (1) identity, (2) manipulability and (3) force.

Therefore the notion of reality is different between mathematicians and philosophers. A mathematician's notion of reality has its own significance in mathematics, particularly when one considers to what extent mathematics is real rather than whether mathematics as a whole is real or not.

2 Mathematical truth

A typical argument against mathematical realism goes as follows:

1. Mathematical statements are about numbers, sets, etc.
2. Numbers, sets, etc. are abstract but not concrete; they do not exist in space-time; they have no causal effect, etc.

3. Therefore they are not real.

There are two assumptions behind this type of argument. One is that mathematical statements are saying something about numbers, sets, etc. Another is that if something is real, it must be concrete; it must exist in space-time; it must have causal effects, etc.

The first assumption has much to do with a model-theoretic conception of the meaning, or model-theoretic interpretation (hereafter referred to as MTI). Roughly speaking, MTI is a doctrine that each well-formed expression in a theory should be associated with something that belongs to the intended model. Names and singular terms are associated with individual objects, predicates with relations, and sentences with conditions. More formally, a model-theoretic interpretation consists of a set of individual objects D, functions $\mathcal{I}_1 : \mathbf{Term} \to D$, $\mathcal{I}_2^n : \mathbf{Pred}_n \to D^n$, where **Term** is the set of terms, and $\mathbf{Pred}_n$ is the set of n-ary predicates.

Language	Model
terms: $a_1, a_2, \ldots, a_n$	objects: $\mathcal{I}_1(a_1), \mathcal{I}_1(a_2), \ldots, \mathcal{I}_1(a_n)$
n-ary predicate: P	n-ary relation: $\mathcal{I}_2^n(P)$
sentence: $P(a_1, a_2, \ldots, a_n)$	condition: $\langle \mathcal{I}_1(a_1), \mathcal{I}_1(a_2), \ldots, \mathcal{I}_1(a_n) \rangle \in \mathcal{I}_2^n(P)$

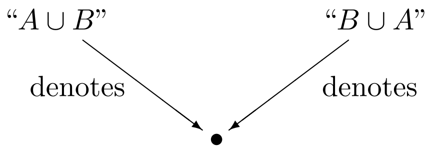
A sentence "$P(a_1, a_2, \ldots, a_n)$" is said to be true if and only if it is the case that

$$\langle \mathcal{I}_1(a_1), \mathcal{I}_1(a_2), \ldots, \mathcal{I}_1(a_n) \rangle \in \mathcal{I}_2^n(P)$$

For example, consider the equation "$A \cup B = B \cup A$." In MTI it is interpreted as indicating the condition that the union of $[\![A]\!]$ and $[\![B]\!]$ is equal to the union of $[\![B]\!]$ and $[\![A]\!]$ ($[\![A]\!]$ and $[\![B]\!]$ are the sets denoted by "A" and "B" respectively).

Language	Model
$A \cup B$	the union of $[\![A]\!]$ and $[\![B]\!]$
$B \cup A$	the union of $[\![B]\!]$ and $[\![A]\!]$
$=$	equality
$A \cup B = B \cup A$	the condition that the union of $[\![A]\!]$ and $[\![B]\!]$ is equal to the union of $[\![B]\!]$ and $[\![A]\!]$

Another way to describe the situation is by saying that the terms "$A \cup B$" and "$B \cup A$" denote the same object.



In MTI, it is necessary that there should be a set denoted by both "$A \cup B$" and "$B \cup A$," but independent of these expressions. Here one views sentences as declarative or descriptive: they mention some objects, describe their conditions, and are true or false depending on whether they are describing the state of these objects correctly or not.

Most arguments around realism in mathematics, whether for or against, implicitly presuppose MTI. Antirealists claim that mathematics is not real because there are no models for mathematical theories anywhere in the real world, while realists claim that the model exists in the realm of forms, or in our thoughts, which are real in some sense. They both take it for granted that in order to claim the reality of mathematics, we should seek out real entities outside mathematics to which mathematical statements refer.

Is this really what happens in mathematics? Modern mathematics does not seem very much to care about what lies outside of the mathematical language, particularly not about what mathematical terms may mean. In fact, the axiomatic method is one of the most remarkable features of modern mathematics. If one asserts that such-and-such is the case, it implies only that there is a formal proof of it, and nothing more. Therefore, whatever a mathematical expression may refer to is entirely irrelevant for mathematics.[1] If so, why do many mathematicians claim the reality of mathematics?

Let me draw an analogy. Suppose someone says "Cars must keep to the left." This statement is true in such countries as the U. K., Australia and Japan, and false in many other countries. Whether it is true or false depends on the traffic law of each country. So the statement is not about a particular situation of things, but about the system of law. The law itself is neither true nor false.

The law is a result of our language activity and therefore not independent of us. It is not a physical entity that exists in space-time. Nor is it a psychological entity that exists someone's mind, since if it is so, it will be subjective, but the very nature of the law requires it to be objective.

The same thing can be said about mathematics. Syntactical rules, definitions, axioms and inference rules are neither true nor false. They are not descriptions of things, but prescriptions for constructing terms, formulas and deductions. However, we can make true statements about the formal system they determine.

It is important to distinguish formulas in an axiomatic system from statements about it. For example, $(xy)^{-1} = y^{-1}x^{-1}$ is not a mathematical statement, but simply a formula. "For every group G, $(xy)^{-1} = y^{-1}x^{-1}$ is deducible for any $x, y \in G$" is a mathematical statement that is true. However, this distinction is only relative, and not a kind of distinction between object language and metalanguage. The latter statement is also within the mathematical language, which is in turn an object of other mathematical statements such as "It is provable that for every group G, $(xy)^{-1} = y^{-1}x^{-1}$ is deducible for any $x, y \in G$."

In fact, there is no clear distinction between object language and metalanguage in mathematics. A statement *about* mathematics is often a statement *within* mathematics. In other words, the mathematical language provides mathematics with objects, and where there is no language, there are no mathematical objects either.

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[1] It may help mathematicians to get some insight, though.

Mathematical objects are products of linguistic activities. However, it does not follow from this that mathematical objects are simply meaningless symbols. In the next chapter we will consider the nature of mathematical objects.

Before proceeding to the next chapter, I should briefly mention semantics for formal systems. Model-theoretic semantics is an almost mundane part of logic. However, it is not really a theory of meanings, but a theory of translation —— from logic into mathematics, in particular into set theory, algebra, topology or category theory. They are also formal systems determined by their own axioms and inference rules. Moreover the means of translation are also given in axiomatic manner. Therefore, model-theoretic semantics does not establish semantic relation between language and things. This fact does not sully the significance of finding a translation from a logic into a mathematical system or structure. Such translation is not only an interesting — sometimes even amazing — mathematical discovery in itself, but also may have epistemic values. For example, Kripke frames are a useful tool in examining whether a formula is provable or not, and so is denotational semantics in examining whether a program is correct or not.

3 Mathematical objects

It is obvious that mathematics does not deal with concrete objects. Nor does it with empty symbols. Mathematical symbols do have meanings, but not in the way words of ordinary language do. For example, a name such as "Caesar" or "Romans" may have Caesar or the set of Romans as their meaning. Symbols in mathematics do not have such concrete objects or even sets of objects as their meaning. They represent functions in the symbol manipulating system, and their functions are determined by syntax, axioms and inference rules.

Consider a simple example from group theory. A group is any quadraple $\langle G, \cdot, \epsilon, ^{-1}\rangle$, where G is a non-empty set, $\cdot$ is a binary operation on G, ϵ is an element of G (called unit), and $^{-1}$ is a unary operation on G satisfying the following conditions: for all $x, y, z \in G$, (A1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

(A2) $x \cdot \epsilon = x = \epsilon \cdot x$,

(A3) $x \cdot x^{-1} = \epsilon = x^{-1} \cdot x$.

We write xy for $x \cdot y$. Owing to (1), parentheses can be omitted.

It follows immediately from (1)-(3) that:

(F1) for all $x, y \in G$, $(xy)^{-1} = y^{-1}x^{-1}$.

(F2) for all $x, y \in G$, if $xy = \epsilon = yx$, then $x^{-1} = y$ and $y^{-1} = x$.

Now let $\mathbf{G} = \langle G, \cdot, \epsilon, ^{-1} \rangle$ be a group satisfying the following conditions.[2]

(GA1) $a, b, c \in G$ where a, b and c are mutually distinct.

(GA2) for all $x \in G$, there are some $z_1, z_2, \ldots, z_n \in \{a, b, c\} (n \geq 0)$ such that

$$x = \epsilon z_1 z_2 \ldots z_n.$$

(GA3) $aa = bb = cc = \epsilon$.

(GA4) $ab = bc = ca$.

(GA5) $ac = cb = ba$.

Since we do not know what a, b and c stand for, we can see elements of $\mathbf{G}$ such as ab, $cabaa$, etc. as mere strings of symbols (we call them *terms*), and the axioms as rules for term rewriting.[3] Let us see what can be said about this formal system.

It follows that:

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[2] $\mathbf{G}$ is an axiomatization of the groups of permutations on lists of three elements.

[3] Cf. [5].

(GF1) $a = a^{-1}, \quad b = b^{-1}, \quad c = c^{-1}$.

(GF2) for all $z_1, z_2, \ldots, z_n \in \{a, b, c\} (n \geq 1)$, $(z_1 z_2 \ldots z_n)^{-1} = z_n \ldots z_2 z_1$.

(GF3) $G = \{a, b, c, ab, ac, \epsilon\}$.

(GF3) may not seem so obvious but it is easy to prove by induction on the length of terms.

For example, Let t be $ab(bac)^{-1}$. Then,

$t = ab\underline{(bac)^{-1}} = abc^{-1}\underline{(ba)^{-1}} = ab\underline{c^{-1}}a^{-1}b^{-1} = abc\underline{a^{-1}}b^{-1} = abca\underline{b^{-1}} = \underline{ab}cab = b\underline{cc}ab = \underline{ba}b = c\underline{bb} = c$

So term t can be viewed as having the term c as its value. The terms which appear in the course of the rewriting of t (including t itself) may also be values of t. Therefore we could not speak of *the* value of t. Fortunately, however, in **G**, every term is equal to just one of ϵ, a, b, c, ab and ac as stated in (GF3), and these are among the simplest of such sets of terms. We will call these terms *normal forms.* If $t = v$ and v is a normal form, v is called the *value* of t. Also we will say that a term t *reduces to* t' if $t = t'$ and t' is not longer than t.

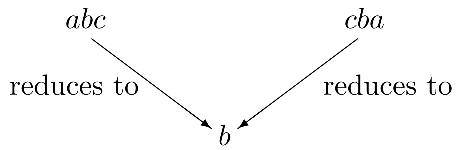
The following conditions are equivalent for every $t, t' \in G$:

(1) $t = t'$,

(2) t and t' has the same value,

(3) t and t' reduce to the same term.

For example $abc = bcc = b$, and $cba = baa = b$. So we can infer $abc = cba$.



Thus a term of **G** has two aspects. One is the value, which determines equality between terms. Another is the form (the configuration of symbols), which indicates how the term is constructed as well as how to calculate its value. The process in which we confirm the equality of *abc* and *cba* is indicated in their composition, and if one knows the syntax, axioms, and rules of **G**, one can construct a proof of the equality. These two aspects resemble Frege's *Bedeutung* and *Sinn*, but we have to be aware that t's value is also a term of **G** and not independent of **G**. The relation between a term and its value is not semantic but proof-theoretic or computational. When we talk of the value of a term t, we really talk of a procedure to deduce $t = v$ (a proof or a construction of $t = v$) in **G** for some $v \in N$.

Moreover, a term $g \in G$ can also be viewed as a function, since for all $x \in G$, $gx \in G$. Here we think of g as a procedure that is applied to arbitrary $x \in G$ and produces the value of gx. Among such functions, some are of interest. For example, define $\alpha := ab$ and $\beta := ac$. Then α is a "one step backward" operator and β a "one step forward" operator, i.e.,

$$\alpha(a) = c \qquad \alpha(b) = a \qquad \alpha(c) = b$$
$$\beta(a) = b \qquad \beta(b) = c \qquad \beta(c) = a$$

It is obvious that $\alpha\alpha = \beta$, $\beta\beta = \alpha$, $\alpha\alpha\alpha = \alpha\beta = \beta\beta\beta = \beta\alpha = \epsilon$, etc. These facts are useful when we carry out calculation. For example, suppose we have to calculate on *acbcacbcababc*. Recalling that $ab = bc = ca$ and $ac = cb = ba$,

$$\begin{aligned} acbcacbcabcac &= \beta\alpha\beta\alpha\alpha\alpha c \\ &= \epsilon\epsilon\beta c \\ &= a \end{aligned}$$

α and β are abstractions which result from neglecting the forms of terms and focusing only on their effects on the atomic terms a, b and c next to them. This example shows that each term has particular functions in particular contexts. Like combinators in lambda calculus, they are names for procedures or programs described in **G**'s terms. And just as programs are meaningful to its interpreter, so are terms of **G** to those who are working at **G**. However, to those who are not familiar with **G**'s axioms or rules, it is no wonder these terms appear meaningless.

The example of **G** shows that even if each atom of language is devoid of meaning, every term of a formal system is not empty. It has its own value as well as functions in particular contexts. In order to know the value and the functions of a term we should calculate it. The way of calculation is shown in its form, together with the syntax, axioms and inference rules of the system. In short, a term expresses the procedures we have to carry on. The same thing can be said about other mathematical objects such as formulas or proofs.

It is such procedures that mathematics is really about. Mathematicians do not deal with mere empty symbols, any more than chess players play with chess pieces. Chess players are playing chess, and the essence of playing chess is to combine possible moves into effective strategies. Similarly, mathematicians combine symbols according to rules into significant terms, formulas and proofs. Once they have been constructed, they can be used to carry on the construction further.

4 Criteria for reality

In this section, we will examine the second assumption of anti-realist arguments that in order for something to be real, it must be concrete; it must exists in space-time; it must have causal effects, etc. These are criteria for what I called substantial reality. It is obvious that the objects of mathematics do not meet these criteria. However, mathematicians seem to be adopting a weaker conception of reality which I called functional. For example, Georg Cantor wrote

> First, we may regard the integers as actual in so far as, on the basis of definitions, they occupy an entirely determinate place in our understanding, are well distinguished from all other parts of our thought, and stand to them in determinate relationships, and thus modify the substance of our mind in a determinate way [...]. ([2], p. 895)

> [concept of mathematics] must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. [7] In particular, in the introduction of new numbers it is only obligated to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to the older numbers that they can in any given instance be precisely distinguished. As soon as a number satisfies all these conditions it can and must be regarded in mathematics as existent and real. ([2], p. 896)

Cantor's criteria for reality include identity, relationships with other things, and effects on other terms and on our activities.[4]. He then claims that they are existent and real in our mind. This is probably because he also felt the need to find out the references of mathematical expressions independent of mathematical language, i.e., he also presupposed a sort of model-theoretic interpretation for mathematical language.

Once we have done away with the model-theoretic view, we no longer have to look for the references of mathematical expressions. In addition, as we have already seen, identity in a formal system is determined only by means of its syntax, axioms and rules. Whether $x = y$ holds or not depends only on whether there is a proof of it or not, which has nothing to do with our state of mind.

I want to reformulate the criteria without referring to any psychological entity. I include the followings in the list of what seems to be mathematicians' criteria for reality: (1) **identity**, (2) **manipulability**, and (3) **force**. I do not claim these to be a necessary and sufficient set of conditions for reality. Nor do I claim a novelty for them. I only hope that they are natural requirements that we expect every real object to fulfill. Let me explain these in order.

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[4] Recall also the quotation from Abelson and Sussman [1] in the begining of the article.

(1) **Identity** of something means its having clear enough characteristics to be qualified as a member of some established category or classification, and to be identified with itself and distinguished from any other object in the category.

For example, we cannot identify the smallest natural number that cannot be designated by any English expression less than twenty words long. Therefore the number is not real. On the other hand, we can identify the smallest natural number that can be designated by some English expression less than twenty words long. It is, of course, zero.

In mathematics, the way to identify something is contained in its definition, and identity is to be proved formally. For example, to identify two functions $f : A \to B$ and $g : C \to D$, it is necessary to prove $A = C$, $B = D$, and for all $x \in A$, $f(x) = g(x)$.

(2) Something is **manipulable** when we can manipulate it according to definite procedures that yield certain (not necessarily deterministic) results when applied to any object in the same category. Here I use the word "manipulate" in a broader sense. For example, construction, application, calculation, detection, measurement etc. are kinds of manipulation.[5]

One important form of manipulation here is making an assertion or judgement about something. Judgement requires some justification for making it. The proposition that a certain apple is sour needs no justification, while if you are to judge so, you need to eat it, to test its acidity, or perhaps to observe someone eat it and frown. There are a great variety of procedures that justify judgements, but in mathematics, there is only one kind: proof.

Manipulability in my sense does not imply that we can do anything we like; on the contrary, manipulations should be understood as quite restricted. The restriction may be imposed either by physical conditions or by conventions. In either case, the stricter the restriction is, the more real the result feels.

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[5] Examples of manipulation which add to reality are abundant in physics, chemistry or medical science.

Axioms and rules of mathematics are among the strictest forms of restriction, because if we break them we can no longer carry on mathematical studies. Therefore mathematics is not a product of our imagination. It is this strictness of mathematics that made Bertrand Russell write:

> Arithmetic must be discovered in just the same sense in which Columbus discovered the West Indies, and we no more create numbers than he created the Indians. The number 2 is not purely mental, but is an entity which may be though *of*. Whatever can be thought of has being, and its being is a precondition, not a result, of its being thought of. ([3], §472)

(3) **Force** is the disposition to invoke in something an action or a change of state, or to prevent it from making an action or changing its state. Again this process need not be deterministic or causal. A typical example of a force that may not be causal is imperative force, i.e., that of (or due to) laws, rules, orders, contracts, appointment, etc. Like the gravity of the earth or a dose of medicine, they have certain effects on us and can throw us into or out of an action.

The intensity of the force that a law or a rule has may differ from person to person, probably depending on one's nature or on how one has been conditioned so far. This difference may account for the fact that many mathematicians are realistic about mathematical objects, while many philosophers are not.

Another example is the force of plans, methods or strategies. For example, think of chess strategies. If a player knows an effective strategy that can be applied to the present situation, he will employ it. So chess strategies have a great effect on chess players. Mathematical objects have this kind of force. Once we have constructed an object, it facilitates further mathematical acivities. For example, if it had not been for a definition of the primes, it would not have been proved that there are an infinite number of primes.

When one employs these criteria, the reality in question is subjective and empirical. It comes in differing degrees from person to person, and therefore different mathematical objects have different degrees of reality. For example, the power set of $\{0,1\}$ is real for every mathematician, while the power set of the set of all natural numbers is not. The question about reality in mathematics is about to what extent mathematics is real, not about whether mathematics as a whole is real or not.

The word "real" in ordinary language has a wide range of meanings. Philosophers and mathematicians are talking about different meanings of the word. So philosophers do not have to feel offended when mathematicians insist on the reality of mathematical objects, and vice versa.

Some may be afraid that the criteria above are too weak and that almost everything will be qualified as real. I do not insist on reality for every procedure. For example, consider a chess strategy that begins with "Take the opponent's queen at the opening of the game, and then..." The most likely response will be "But how can I take the opponent's queen at the opening?"

It is meaningful to ask whether a procedure is feasible, realizable or effective. These notions have something to do with the notion of reality as used by mathematicians. It is useless to dispute whether mathematics as a whole is real or not, for, as we have seen, it is a mere matter of definition of the word. A more significant question is to what extent mathematics is real. For it is on this question that disagreement between mathematicians arises. Few mathematicians are sheer anti-realists who do not admit the reality of natural numbers, or full-blooded realists who admit the existence of the set of everything. Disputes over reality in mathematics are about where to stop. Finitism, intuitionism or Platonism; predicative or impredicative analysis; first-order, second-order or higher-order predicate logic; ZF, ZFC or more; at which level of the arithmetical hierarchy; typed or untyped lambda calculus; context-sensitive or phrase structure grammar; etc.

5 Conclusion

Mathematical objects are essentially different from physical or everyday objects. Therefore, when mathematicians say they are real, a different meaning of reality is intended. Rather than blaming mathematicians for being careless in the use of the word, it is useful to consider in what sense mathematical objects are real as well as what the nature of mathematical objects is.

My conclusion is that objects of mathematics are procedures expressed by a formal system, not something independent of or referred to by mathematical language. Nevertheless they can be regarded as real in some weaker sense.

Whether you call procedures real or not may be only a matter of different viewpoints. However, discoveries of different viewpoints have been triggering important advances in mathematics. In particular, to view a procedure as an object gave rise to lambda calculus, related programming language, and model theory such as denotational semantics[6]. In this sense also, realism in mathematics has certain significance.

References

[1] H. Abelson and G. J. Sussman. *Structure and Interpretation of Computer Programs*. The MIT Press, Massachusetts, second edition, 1996.

[2] G. Cantor. Foundations of a general theory of manifolds: a mathematico-philosophical investigation into the theory of the infinite. In W. Ewald, editor, *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Vol. 2, pp. 878–920. Oxford University Press, New York, 1883.

[3] B. Russell. *The Principles of Mathematics*. Routledge, London, second edition, 1992.

[6] Cf. [4]. In Japanese, see [6].

[4] J. E. Stoy. *Denotational Semantics: The Scott-Strachey Approach to Programming Language Theory*. The MIT Press, Massachusetts, 1977.

[5] Terese. *Term Rewriting Systems*, Vol. 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 2003.

[6] 横内寛文. 『プログラム意味論』, 情報数学講座, 第 7 巻. 共立出版, 1994.